Maps between curves

Recall from the exercises that a map

$X \to Y$

of two ns, ~~proper~~ curves is ~~surjective~~ finite or constant.

$\ast$ X, Y ns, connected proper curves $/ k = \bar{k}$

Def $\pi : X \to Y$ finite morphism

$\deg(\pi) = [K(X) : K(Y)]$ degree

ramification index at $p \in |X|$

$e_p := \mathrm{ord}_p(\pi^\flat(t))$ for $t \in \mathcal{O}_{Y,\pi(p)}$ local parameter.

if $e_p > 1$, say π is ramified at p and $\pi(p)$ is a branch point

if $e_p = 1$, say π is unramified at p

if $\mathrm{char}(k) \mid e_p$ then p is a wild ramif. pt

if $\mathrm{char}(k) \nmid e_p$ then p is a tame ramif. pt.

Def $\pi : X \to Y$ finite morphism.

$$\pi^* : \mathrm{Div}(Y) \to \mathrm{Div}(X)$$
$$q \mapsto \sum_{p \in \pi^{-1}(q)} e_p [p]$$

pullback of divisors.

Lem $\pi : X \to Y$ finite

$$\pi^* \mathcal{O}_Y(D) \cong \mathcal{O}_X(\pi^* D)$$

Pf (idea) Compute divisor of zeros and poles of $\pi^* \mathcal{O}_X(D)$ $\square$

Def $\pi : X \to Y$ finite
is <u>separable</u> if $K(X)/K(Y)$ is seperable

<u>Proposition</u> $\pi : X \to Y$ finite seperable.

$$0 \to \pi^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0$$

is a SES.

$p(a) = 0$
$p'(a) \neq 0$
$dp(a) = p'(a)\,da = 0$
$\Rightarrow da = 0 \ \forall a \in \Omega_{X/Y}$

<u>Proof</u> Show $\pi^*\Omega_Y \to \Omega_X$ injective.

$K(X)/K(Y)$ seperable $\Rightarrow (\Omega_{X/Y})_{\eta_X} = 0$

$\Rightarrow \pi^*\Omega_Y \to \Omega_X$ nonzero morphism of line bundles

$\Rightarrow \pi^*\Omega_Y \to \Omega_X$ injective. $\square$

<u>Proposition</u> $\pi : X \to Y$ finite separable

(a) $\Omega_{X/Y}$ is a torsion sheaf

$$\mathrm{supp}(\Omega_{X/Y}) = \{ p \in |X| \mid \pi \text{ ramified @ } p \}$$

(b) for all $p \in |X|$ $(\Omega_{X/Y})_p$ is a principal $\mathcal{O}_{X,p}$ module of finite length and

$$\mathrm{length}_{\mathcal{O}_{X,p}}(\Omega_{X/Y})_p = \mathrm{ord}_p\left(\frac{d\pi^b(t)}{ds}\right)$$

where $t \in \mathcal{O}_{Y,\pi(p)}$, $s \in \mathcal{O}_{X,p}$ are local parameters.

(c) π tamely ramified at p

$$\Rightarrow \mathrm{length}_{\mathcal{O}_{X,p}}(\Omega_{X/Y})_p = e_p - 1$$

π wildly ramified at p

$$\Rightarrow \mathrm{length}_{\mathcal{O}_{X,p}}(\Omega_{X/Y})_p > e_p - 1$$

<u>proof</u> local calculation –

(a) SES (*) $\Rightarrow$ $\Omega_{X/Y}$ torsion.

$(\Omega_{X/Y})_p = 0$ iff $\pi^b(dt) \in \Omega_{X,p} = \mathcal{O}_{X,p} ds$ is a generator

iff $\pi^b(t) = su$ is a local parameter.

$\pi^b(dt) = d(\pi^b(t))$, $\pi^b(t) = s^n u$

$d(\pi^b(t)) = d(s^n u) = n s^{n-1} u\, ds + \underbrace{s^n du}_{=0}$

$(*)_p$: $0 \to \mathcal{O}_{X,p}\pi^b(dt) \to \mathcal{O}_{X,p} ds \to (\Omega_{X/Y})_p \to 0$

$\Rightarrow (\Omega_{X/Y})_p \cong \mathcal{O}_{X,p}/(ns^{n-1})$

$\Rightarrow$ (b), (c) $\square$

<u>Def</u> $\pi : X \to Y$ finite seperable.

The <u>ramification divisor</u> of π is

$$\mathrm{Ram}(\pi) = \sum_{p \in |X|} \mathrm{length}_{\mathcal{O}_{X,p}} (\Omega_{X/Y})_p \; p \quad \in \mathrm{Div}(X)$$

<u>effective</u>

<u>Lemma</u> $\pi : X \to Y$ finite seperable

$$\Omega_X \cong \pi^* \Omega_Y \otimes \mathcal{O}_X(\mathrm{Ram}(\pi))$$

<u>Proof</u> $(*) \otimes \Omega_X^\vee$:

$$0 \to \pi^* \Omega_Y \otimes \Omega_X^\vee \to \mathcal{O}_X \to \mathcal{O}_{\mathrm{Ram}(\pi)} \to$$

Here interpret $\mathrm{Ram}(\pi) \subset X$ as the subscheme $\coprod_{p \in |X|} \mathrm{Spec}(\mathcal{O}_{X,p}/(ns^{n-1}))$

zero dimensional <u>non-redu</u>

$\Rightarrow \pi^* \Omega_Y \otimes \Omega_X^\vee \cong \mathcal{O}_X(-\mathrm{Ram}(\pi))$

$\Rightarrow$ <u>Thm</u> (Riemann – Hurwitz)

$\pi : X \to Y$ finite seperable.

- $2g(X) - 2 = \deg(\pi)(2g(Y) - 2) + \deg(\mathrm{Ram}(\pi$

$= \deg(\pi)(2g(Y) - 2) + \sum_{p \in |X|} (e_p - 1)$ if π tame ramified

Def $\mathrm{char}(k) = p \rightsquigarrow$ $\mathrm{Frob}: k \to k$ $a \mapsto a^p$
ring homom

S scheme $/k$.
The geometric Frobenius of S is the morphism obtained by base change:

$$\begin{tikzcd}
S_\Phi \arrow[r, "\Phi"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dr, bend right=60, "\text{structure morphism}"'] & S \arrow[d] \\
\mathrm{Spec}(k) \arrow[r, "\Phi = \mathrm{Spec}(\mathrm{Frob})"'] & \mathrm{Spec}(k)
\end{tikzcd}$$

Example

$\Phi : Y_\Phi \to Y$ is finite of degree p

check it!

$$K(Y_\Phi) \cong k \otimes_{\Phi, k} K(X) \cong \{a \in \overline{K(Y)} \mid a^p \in K(Y)\} = K(Y)^{1/p}$$

$y \in |Y_\Phi|$ $\quad \Phi^\flat : \mathcal{O}_{Y, \Phi(y)} \to \mathcal{O}_{Y_\Phi, y}$
$a \longmapsto a^p$

embed the fields as subfields of $\overline{K(Y)}$ or $K(Y)^{1/p}$.

$\Rightarrow$ Φ ramified everywhere with ramification index p

Prop. $\pi : X \to Y$ finite purely inseperable
$\Rightarrow X \cong Y$ over $\mathrm{Spec}(\mathbb{Z})$ (not $\mathrm{Spec}(k)$!) and π is a composition of geometric Frobenius morphisms. In particular $g(X) = g(Y)$

Proof sketch: $\deg(\pi) = p^r \Rightarrow K(X)^{p^r} \subseteq K(Y)$
$\Leftrightarrow K(X) \subseteq K(Y)^{1/p^r} = K(Y_\Phi)$

$\deg((\Phi_Y)^r) = p^r$

same degree $\Rightarrow X \cong Y_\Phi$

$+$ equivalence of categories

$\longrightarrow$ can factor any $\pi : X \to Y$ (p.i. $X \to X' \to Y$ sep) $\rightsquigarrow g(X) = g(X') \leftrightarrow g(Y)$. $\square$

Def X, $g(X) \geq 1$ is hyperelliptic if there is a degree 2 morphism $\pi : X \xrightarrow{2:1} \mathbb{P}^1$

Ex. $g(X) = 2$. $X \to \mathbb{P}(H^0(X, \omega_X))$ is such a cover.

$RH \Rightarrow$ Lemma $\deg(\mathrm{Ram}(\pi)) = 2g+2$

Prop X hyperelliptic iff there is a line bundle $\mathcal{L}$ on X with $h^0(\mathcal{L}) = 2 = \deg(\mathcal{L})$.

Proof Clearly for such an $\mathcal{L}$ $X \to \mathbb{P}(H^0(\mathcal{L}))$ is the desired cover.

Conversely set $\mathcal{L} := \pi^* \mathcal{O}_{\mathbb{P}^1}(1)$

$\Rightarrow \deg(\mathcal{L}) = 2$ ✓ (exercises)

$h^0(\mathcal{L}) \geq 2$. (needed for $X \to \mathbb{P}(H^0(\mathcal{L}))$)

if $h^0(\mathcal{L}) \geq 3$:

$h^0(\mathcal{L}(-p)) = 3$ or 2 for all $p \in |X|$

if $h^0(\mathcal{L}(-p)) = 3$

$\Rightarrow h^0(\underbrace{\mathcal{L}(-p-q)}_{\deg 0}) = 3, 2 > 1$ ↯

$\Rightarrow h^0(\mathcal{L}(-p)) = 2$ for all $p \in |X|$

$\Rightarrow \pi : X \to \mathbb{P}^1$ injective ↯ □

Ex: $g(X) = 1$ "elliptic" $\Rightarrow X$ hyperelliptic.